Stress distributions
Concrete beam behavior

Satisfy $M_{int} = M$, and $P = 0$
Composite beams

- \( dF = \sigma dA = (E_1 \varepsilon) dz dy \)
- \( dF' = \sigma' dA' = (E_2 \varepsilon') ndz dy \)
- Equating \( dF \) and \( dF' \)
- \( n = E_1 / E_2 \)
- The force in material 1 is
- \( dF = \sigma dA = \sigma' dA' \)
- \( \sigma dz dy = \sigma' ndz dy \)
- \( \sigma = n \sigma' \)

Uncracked concrete beams

(a) \( nA_\theta \)

(b) \( (n-1)A_\theta \)
Stresses in uncracked beam

EXAMPLE 3.1 A rectangular beam has the dimensions (see Fig. 3.2b) \(b = 10 \text{ in.}, \ h = 25 \text{ in.}, \ d = 23 \text{ in.}\), and is reinforced with three No. 8 (No. 25) bars so that \(A_e = 2.37 \text{ in}^2\). The concrete cylinder strength \(f'_c\) is 4000 psi, and the tensile strength in bending (modulus of rupture) is 475 psi. The yield point of the steel \(f_y\) is 60,000 psi, the stress-strain curves of the materials being those of Fig. 1.16. Determine the stresses caused by a bending moment \(M = 45 \text{ ft-kips}\).

**Solution.** With a value \(n = E_y/E_f = 29,000,000/3,600,000 = 8\), one has to add to the rectangular outline an area \((a - 1)A_e = 7 \times 2.37 = 16.59 \text{ in}^2\), disposed as shown on Fig. 3.4, to obtain the uncracked, transformed section. Conventional calculations show that the location of the neutral axis of this section is given by \(\bar{y} = 13.2 \text{ in.}\), and its moment of inertia about this axis is 14,740 \text{ in}^4. For \(M = 45 \text{ ft-kips} = 540,000 \text{ in-lb}\), the concrete compression stress at the top fiber is, from Eq. (3.3),

\[
f_c = \frac{540,000 \times 13.2}{14,740} = 484 \text{ psi}
\]

and, similarly, the concrete tension stress at the bottom fiber is

\[
f_c = \frac{540,000 \times 11.8}{14,740} = 432 \text{ psi}
\]

Since this value is below the given tensile bending strength of the concrete, 475 psi, no tension cracks will form, and calculation by the uncracked, transformed section is justified. The stress in the steel, from Eqs. (1.6) and (3.2), is

\[
f_y = \frac{M_y}{I} = \frac{540,000 \times 9.8}{14,740} = 2870 \text{ psi}
\]

By comparing \(f_c\) and \(f_y\) with the cylinder strength and the yield point respectively, it is seen that at this stage the actual stresses are quite small compared with the available strengths of the two materials.
Stresses on cracked beam

1. Find neutral axis
2. Find I cr
3. Find stresses

\[ M = C j d = \frac{f_c}{2} b k d j d = \frac{f_c}{2} k j b d^2 \]

\[ f_s = \frac{2M}{k j b d^2} \]

\[ \rho = \frac{A_s}{b d} \]

\[ k = \sqrt{\left(\rho n\right)^2 + 2\rho n - \rho n} \]

\[ j d = d - \frac{kd}{3} \]

\[ j = 1 - \frac{k}{3} \]
### Design aids

**Table A.6**

Parameters \( k \) and \( j \) for elastic, cracked section beam analysis, where \( k = \sqrt{2\pi n} \left( \frac{n}{\mu} \right)^{3/2} \) and \( j = 1 - \frac{1}{k} \)

<table>
<thead>
<tr>
<th>( n )</th>
<th>( k )</th>
<th>( j )</th>
<th>( k )</th>
<th>( j )</th>
<th>( k )</th>
<th>( j )</th>
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<td>0.238</td>
<td>0.022</td>
</tr>
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<td>11</td>
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<td>0.265</td>
<td>0.018</td>
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<tr>
<td>12</td>
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<td>0.910</td>
<td>0.287</td>
<td>0.011</td>
<td>0.302</td>
<td>0.009</td>
</tr>
</tbody>
</table>

---

### Example 3.2

The beam of Example 3.1 is subject to a bending moment \( M = 90 \) ft-kips (rather than 45 ft-kips as previously). Calculate the relevant properties and stresses.

**Solution.** If the section were to remain uncracked, the tensile stress in the concrete would now be twice its previous value, that is, 864 psi. Since this exceeds by far the modulus of rupture of the given concrete (475 psi), cracks will have formed and the analysis must be adapted appropriately. Equation (3.5), with the known quantities \( b, n, \) and \( A_s \) inserted, gives

\[
d_k = 7.6 \text{ in., or } k = 7.6 / 23 = 0.33.
\]

From Eq. (3.13), \( j = 1 - 0.33 / 3 = 0.89 \). With these values the steel stress is obtained from Eq. (3.8) as \( f_s = 22,300 \text{ psi, and the maximum concrete stress from Eq. (3.10) as } f_c = 1390 \text{ psi.} \)

Comparing the results with the pertinent values for the same beam when subject to one-half the moment, as previously calculated, one notices that (1) the neutral plane has migrated upward so that its distance from the top fiber has changed from 13.2 to 7.6 in., (2) even though the bending moment has only been doubled, the steel stress has increased from 2870 to 22,300 psi; or about 7.8 times; and the concrete compression stress has increased from 484 to 1390 psi, or about 2.9 times; (3) the moment of inertia of the cracked transformed section is easily computed to be 5.910 in\(^4\), compared with 14.740 in\(^4\) for the uncracked section. This affects the magnitude of the deflection, as discussed in Chapter 6. Thus, it is seen how radical is the influence of the formation of tension cracks on the behavior of reinforced concrete beams.
Ultimate flexural strength

- Steel fails when \( f_s = f_y \)
- Concrete fails when \( \varepsilon_c = \varepsilon_u = 0.003 \)
- Knowing \( c \) need to know: \( C \) and \( \beta \)

![Diagram showing the relationship between stress, strain, and failure criteria in concrete and steel.]

Ultimate flexural strength

Compressive force in a rectangular beam:

\[
C = f_{cw}bc \quad (1)
\]

Average compressive stress:

\[
f_{cw} = \frac{f_c}{b} \quad (2)
\]

Ultimate compressive stress:

\[
f_c = f_{cw}bc \quad (3)
\]

Applying equilibrium:

\[
C = T = x f_l b c = A f_s \quad (4)
\]

Bending moment:

\[
M = T z = A f_s (d - \beta c) \quad (5)
\]

\[
M = C z = x f_b b c (d - \gamma c) \quad (6)
\]
Ultimate flexural strength

Failure initiated by yielding of the steel \( f_s = f_y \) (under-reinforced beam)

Neutral axis depth

\[ c = \frac{A_y f_y}{f_y} \times \frac{P}{b d} \]

Nominal moment capacity

\[ M_n = P f_y b d \left( 1 - \frac{f_y}{f_y} \right) \]

Substitute \( f_y = 0.7 f_y \), \( P = 0.4 P \)

\[ M_n = P f_y b d \left( 1 - 0.5 f_y \right) \]

Over-reinforced beam - compression failure

\[ \varepsilon_s = 0.003 \]

\[ f_s = f_y E_s = f_y E_s \frac{d - c}{c} \]

Solve the previous quadratic equation for \( c \)

Knowing \( c \) substitute above to find

Substitute in Eqs. 5 or 6 to find \( M_n \)

Ultimate flexural strength

Balanced failure condition: Steel yields at the same time as concrete crushes

\[ f_s = f_y \]

\[ \varepsilon_s = \frac{f_y}{E_s} \]

Neutral axis depth

\[ c = \frac{E_s}{E_s + \varepsilon_s} \]

Substitute in

\[ \frac{A_y f_y}{b d} = \frac{A_s f_y}{b d} \]

to obtain

\[ f_s = \frac{f_y}{E_s + \varepsilon_s} \]
Example 3.3

Determine the nominal moment $M_n$ at which the beam of Examples 3.1 and 3.2 will fail.

**Solution.** For this beam the reinforcement ratio $\rho = A_s / bd = 2.37 / (10 \times 23) = 0.0103$. The balanced reinforcement ratio is found from Eq. (3.24) to be 0.0284. Since the amount of steel in the beam is less than that which would cause failure by crushing of the concrete, the beam will fail in tension by yielding of the steel. Its nominal moment, from Eq. (3.20b), is

$$M_n = 0.0103 \times 60,000 \times 10 \times 23 \left( 1 - 0.59 \frac{0.0103 \times 60,000}{4000} \right)$$

$$= 2,970,000 \text{ in-lb} = 248 \text{ ft-kips}$$

When the beam reaches $M_n$, the distance to its neutral axis, from Eq. (3.19b), is

$$c = \frac{0.0103 \times 60,000 \times 23}{0.72 \times 4000} = 4.94$$

---

**Ultimate flexural strength**

- **Whitney’s equivalent rectangular stress block**

![Diagram of Whitney's equivalent rectangular stress block](image-url)
Rectangular stress block

Allan's equivalent rectangular stress block.
- Maintain resistance in the same location as found from tests
  \[ \frac{c}{d} = \frac{a}{b} \Rightarrow \frac{a}{b} = 2:3 \]
- \( C \) from tests = \( C \) equivalent
- \( \frac{a}{c} \cdot \frac{b}{d} = \frac{F_{\text{lab}}}{C} \)
- \( \frac{a}{c} \cdot \frac{b}{d} = \frac{F_{\text{lab}}}{C} \)
  \[ \frac{a}{c} = \frac{2}{3}, \frac{b}{d} = \frac{3}{2} \]
  \[ \frac{a}{c} = \frac{2}{3}, \frac{b}{d} = \frac{3}{2} \]
  \[ a = 2b, c = \frac{3b}{2} \]
  \[ a = 2b, c = \frac{3b}{2} \]
  
  Since \( \frac{a}{c} = 0.85 \)
  \[ C = 0.85F_{\text{lab}} \]

<table>
<thead>
<tr>
<th>Concrete stress block parameters</th>
</tr>
</thead>
<tbody>
<tr>
<td>( f_c, \text{ psi} )</td>
</tr>
<tr>
<td>0-4000</td>
</tr>
<tr>
<td>( a )</td>
</tr>
<tr>
<td>( b )</td>
</tr>
<tr>
<td>( \rho_i = \frac{2b}{b+c} )</td>
</tr>
<tr>
<td>( \beta = \frac{a}{b} )</td>
</tr>
</tbody>
</table>

Rectangular stress block

For \( f_c = 4000 \text{ psi} \):
\[ \beta = 0.85 \]

For \( f_c > 4000 \text{ psi} \):
\[ \beta = 0.65 - 0.05 \left( \frac{f_c - 4000}{1000} \right) \geq 0.65 \]

Lever arm of the equivalent rectangular stress block
\[ z = d - \frac{a}{2} \]

Moment capacities
\[ M = Tz = Cz \]

Enhanced failure
\[ f_b = 0.05 \frac{f_c}{\gamma} \frac{E_0}{E_0 + E_1} \]

For \( E_0 = 0.005 \) and \( E_1 = 20 \times 10^6 \text{ psi} \)
\[ f_b = 0.05 \frac{f_c}{\gamma} \frac{E_0}{E_0 + E_1} \]
Moment capacity of beams

Depth of rectangular stress block

\[ a = \frac{f_{yd}}{0.85 f'_c} \]

Nominal moment capacity

\[ M_n = f_{yd} b d^2 \left( 1 - 0.59 \frac{f_{yd}}{f'_c} \right) \]

Let

\[ K = f_{yd} \left( 1 - 0.59 \frac{f_{yd}}{f'_c} \right) \]

\[ s = M_n = K b d^2 \]

Design moment capacity

\[ \phi M_n = s = K b d^2 \]

---

Design aids - resistance factor

**TABLE A.5a**

Flexural resistance factor: \( R = \frac{f_y}{f'_c} \left( 1 - 0.588 \frac{f_y}{f'_c} \right) \)

\( f'_c = 40,000 \) psi

<table>
<thead>
<tr>
<th>( f'_c = 60,000 ) psi</th>
</tr>
</thead>
<tbody>
<tr>
<td>( f'_c, \text{ psi} )</td>
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<tr>
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<tr>
<td>0.0095</td>
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<tr>
<td>0.0100</td>
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</tbody>
</table>
Example 3.4

- Solve the same beam using the rectangular stress block

Solution. The distribution of stresses, internal forces, and strains is as shown in Fig. 3.11. The maximum reinforcement ratio is calculated from Eq. (3.30b) as

\[
\rho_{max} = 0.85 \times 0.85 \times \frac{4000}{60,000} = 0.0303 + 0.004 = 0.0206
\]

and comparison with the actual reinforcement ratio of 0.0103 confirms that the member is underreinforced and will fail by yielding of the steel. The depth of the equivalent stress block is found from the equilibrium condition that \(C = T\). Hence \(0.85\rho_{ab} = A_{f_s}\), or

\[a = 2.37 \times 60,000/(0.85 \times 4000 \times 10) = 4.18\].

The distance to the neutral axis, by definition of the rectangular stress block, is \(c = a/\beta_i = 4.18/0.85 = 4.92\). The nominal moment is

\[M_e = A_{f_s} \left( d - \frac{a}{2} \right) = 2.37 \times 60,000(23 - 2.09) = 2,970,000 \text{ in-lb} = 248 \text{ ft-kips}\]

Calculate the design moment capacity for the beam analyzed in Example 3.4.

Solution. For a distance to the neutral axis of \(c = 4.92\), \(\epsilon_r = 0.003(23 - 4.92) = 0.011\) from Eq. (3.28), \(\epsilon_r > 0.005\), so \(\phi = 0.90\) and the design capacity is

\[dM_e = 0.9 \times 248 = 223 \text{ ft-kips}\]

Limiting reinforcement ratios

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<th>(f_s) psi</th>
<th>(f_y) psi</th>
<th>(\rho_{t} = \frac{\sigma}{f_y})</th>
<th>(\rho_{t} = 0.005)</th>
<th>(\rho_{t} = 0.004)</th>
<th>(\rho_{max} = \frac{200}{f_y})</th>
<th>(\rho_{max} = \frac{3.1}{f_y})</th>
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<td>0.0137</td>
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<tr>
<td>11,000</td>
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<td>0.0414</td>
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<td>0.0474</td>
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<td>0.0137</td>
</tr>
<tr>
<td>18,000</td>
<td>17,000</td>
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<td>0.0414</td>
<td>0.0474</td>
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<td>19,000</td>
<td>18,000</td>
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<td>0.0414</td>
<td>0.0474</td>
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<td>0.0535</td>
<td>0.0137</td>
</tr>
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</table>
ACI strength reduction factors

- **ACI (9.3.2)**
  - **Tension-controlled failure**
    - When $\varepsilon_c = 0.003$, $\varepsilon_s > 0.005$, So $\phi = 0.9$
  - **Compression-controlled failure**
    - When $\varepsilon_c = 0.003$, $\varepsilon_s < \varepsilon_y = 0.002$, So $\phi = 0.65$
  - **Transition-controlled failure**
    - When $\varepsilon_c = 0.003$, $\varepsilon_y < \varepsilon_s < 0.005$
    - So $\phi = A_1 + B_1 \varepsilon_t$

$$A_1 = \frac{0.00325 - 0.9\varepsilon_y}{0.005 - \varepsilon_y} \hspace{1cm} B_1 = \frac{0.25}{0.005 - \varepsilon_y}$$
ACI transition-controlled failure

- Reason for limiting $\varepsilon_t > 0.004$ in tension-controlled failure (ACI 10.3.5)
  \[ \rho = \frac{A_t}{bd} \]
  \[ \rho_{bal} = \frac{0.85\beta_1 f'_c}{f_y} \frac{0.003}{0.003 + \varepsilon_t} \]

- In 2002 ACI code
  - Limit $\rho$ to $\rho_{max}$ to $0.75\rho_{bal}$
  - Results in $\varepsilon_t = 0.00376$, so limit $\varepsilon_t$ to 0.004
  - End up with $\rho_{max} < 0.75\rho_{bal}$

ACI tension-controlled failure

- From \[ \frac{c}{d_t} = \frac{0.003}{0.003 + \varepsilon_t} \]
  - For $\varepsilon_t = 0.004$, $c/ d_t = 3/7$
  - And $c_{max} = 3/7 d_t$
  - So $a_{max} = \beta_1 c_{max} = 3/7\beta_1 d_t$
  - So $\rho_{max} = 0.364 \frac{\beta_1 f'c/ fy}{(d_t/ d)}$
**ACI** tension-controlled failure

- From \[ \frac{c}{d_t} = \frac{0.003}{0.003 + \varepsilon_t} \]

- For \( \varepsilon_t = 0.005, \frac{c}{d_t} = \frac{3}{8} \)

- And \( c_{\text{max}} = \frac{3}{8} d_t \)

- So \( a_{\text{max}} = \beta_1 c_{\text{max}} = \frac{3}{8} \beta_1 d_t \)

- So \( \rho_{\text{max}} = 0.319 \beta_1 \frac{f'c}{f_y} (d_t/d) \)

**ACI** minimum reinforcement

- **ACI** 10.5.1 requires that

\[
A_{s,\text{min}} = \frac{3\sqrt{f_c}}{f_y} b_w d \geq \frac{200}{f_y} b_w d
\]

**TABLE A2–3** inms and \( \rho_{\text{cr}} \) for Common Grades of Steel and Compressive Strength of Concrete (Single Layer of Steel, i.e., \( d = d_t \))

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<th>( f_{\text{cr}} = 0.005 )</th>
<th>( f_{\text{cr}} = 0.005 )</th>
<th>( f_{\text{cr}} = 0.005 )</th>
<th>( f_{\text{cr}} = 0.005 )</th>
<th>( \rho_{\text{cr}} )</th>
</tr>
</thead>
<tbody>
<tr>
<td>40,000</td>
<td>0.0032</td>
<td>0.0031</td>
<td>0.0064</td>
<td>0.0364</td>
<td>0.83</td>
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<td>60,000</td>
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<td>0.0207</td>
<td>0.0243</td>
<td>0.0243</td>
<td>0.61</td>
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<td>75,000</td>
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<td>0.0166</td>
<td>0.0194</td>
<td>0.0194</td>
<td>0.60</td>
</tr>
</tbody>
</table>

**TABLE A2–4** Minimum Steel Ratio \( (\rho_{\text{min}}) \)

<table>
<thead>
<tr>
<th>( f_y ) (psi)</th>
<th>( f_{\text{cr}} = 3,000 ) psi</th>
<th>( f_{\text{cr}} = 4,000 ) psi</th>
<th>( f_{\text{cr}} = 5,000 ) psi</th>
<th>( f_{\text{cr}} = 6,000 ) psi</th>
</tr>
</thead>
<tbody>
<tr>
<td>40,000</td>
<td>0.0050</td>
<td>0.0050</td>
<td>0.0053</td>
<td>0.0058</td>
</tr>
<tr>
<td>60,000</td>
<td>0.0033</td>
<td>0.0033</td>
<td>0.0035</td>
<td>0.0039</td>
</tr>
<tr>
<td>75,000</td>
<td>0.0027</td>
<td>0.0027</td>
<td>0.0028</td>
<td>0.0031</td>
</tr>
</tbody>
</table>
Example - analysis

Use Method 1 to determine the design resisting moment, $M_R$, of the reinforced concrete beam section shown below. $f'_c = 4$ ksi, and $f'_s = 40$ ksi. The reinforcement is 3 #9 bars, $A_s = 3.00$ in$^2$.

Solution

Using the steps of Figure 2–39:

Step 1 Find the steel ratio, $\rho$:

$$\rho = \frac{A_s}{bd} = \frac{3}{10 \times 22} = 0.0136$$

Example - analysis

Step 1 From Example 2–6:

$$\rho = 0.0136 > \rho_{\text{min}} = 0.0050 \quad \therefore \text{ok}$$

From Table A2–3 $\rho_{\text{max}} = 0.0310 > 0.0136 \quad \therefore \text{ok}$

Step 2 Using $\rho = 0.0136$, $f'_s = 40$ ksi, and $f'_c = 4$ ksi, obtain the resistance coefficient, $R$, from Table A2–5b:

$$R = 450 \text{ psi}, \phi = 0.90$$

Step 3 Because the beam has a single layer of reinforcement:

$$M_R = \frac{bd^2R}{12,000} = \frac{10 \times 22^2 \times 450}{12,000}$$

$$M_R = 182 \text{ ft-k}$$
Design of R/C beams

- ACI 9.5.2.1 minimum span/depth ratios for beams

![Diagram of R/C beam configurations and span/depth ratios](image)

Design of R/C slabs

- ACI 9.5.2.1 minimum span/depth ratios for one-way slabs

![Diagram of R/C slab configurations and span/depth ratios](image)
Design of R/C beams

- Selection of width
- ACI 7.6.1, 7.6.2, and 3.3.2
- Minimum space for single layer bars
  - $s_{\text{min}} = \text{largest of } (d_b \text{ or } 1 \text{ in. or } \frac{3}{4} \text{ max aggregate size})$

Design of R/C beams

- Minimum space for multiple layers of bars
  - Bars in upper layer placed directly above bottom layer
  - Clear distance between layers $> 1$ in.
  - Also satisfy single layer requirements
Design of R/C beams

- **Minimum cover**
  - Cast in place concrete - protection
    - ¾ in. for slabs and 1 ½ in. for beams and columns
  - Exposed to weather or in contact with soil
    - Cover > 2 in.
    - Concrete cast directly on ground - cover >= 3 in.

- **Minimum width**
  - Usually #3 or #4 are used for stirrups
  - Minimum cover for bars in beam is 1.5 in.
  - $b_{\text{min}} = 2 \times 1.5 \text{ in.} + 2 \times 1/2 \text{ in.} + 4 \times 1 \text{ in.} + 3 \times 1 \text{ in.} = 11 \text{ in.}$
Design example 1

Figure 2–47a shows the partial framing plan of a beam-girder reinforced concrete floor system. The slab is 6 in. thick, and is subjected to a superimposed dead load of 30 psf. The floor live load is 100 psf. Beam B-2 has a width of 12 in. (b = 12 in.), and a total depth of 30 in. (including the slab thickness). Determine the steel required at Section 1.1. Use the ACI Code coefficients to calculate moments. Assume that the beam end is integral with the column. Use $f'_c = 4$ ksi, $f_y = 60$ ksi, and assume that the unit weight of concrete is 150 pcf. Stirrups are #3 bars.

Solution

Step 1  Before calculating the moments at the selected location, we must determine the floor loads:

- Weight of slab = $150 \times \left( \frac{h}{12} \right) = 75$ psf
- Superimposed dead load = 30 psf
- Total dead load = 105 psf
- Live load = 100 psf

The tributary width for beam B-2 is 15’-0”; therefore, the uniform dead and live loads are:

$$w_D = \frac{105 \times 15}{1000} + \frac{150 \times 12 \times 24}{12 \times 12 \times 1000} = 1.88 \text{ kip/ft}$$

Design example 1

$$w_L = \frac{100 \times 15}{1000} = 1.5 \text{ kip/ft}$$

Note: Reduction of live load is neglected here.

$$w_w = 1.2w_L + 1.6w_{wL} = 1.2 \times 1.88 + 1.6 \times 1.5 = 4.65 \text{ kip/ft}$$

The beam clear span $\ell_n = 30 \text{ ft} - (0.5 \text{ ft} + 0.5 \text{ ft}) = 29 \text{ ft}$
Design example 1

Figure 2-47b shows the moment using the A3.0 coefficients from Table A2-1 for an exterior beam. Because the problem requires designing the reinforcement at section 1-1:

\[ M_{xy} = \frac{wL^2}{18} \times \frac{4.65 \times 391}{10} = 391 \text{ k-ft} \]

Step 2 Assuming the distance (y) from the edge of the beam in tension to the center of tensile steel is 2.5 in:

\[ d = h - y = 30 \text{ in.} - 2.5 \text{ in.} = 27.5 \text{ in.} \]

Step 3 The required resistance coefficient, \( R \), is:

\[ R = \frac{12,000M_y}{h d^2} = \frac{12,000 \times 391}{12 \times 27.5^2} \]

\[ R = 517 \text{ psi} \]

Step 4

\[ \sigma_y = 4 \text{ ksi} \quad \rightarrow \quad \text{Table A2-60} \quad \rightarrow \quad \rho = 0.0106 \]

\[ f_y = 60 \text{ksi} \]

Note that \( \rho = 0.0106 \) corresponding to \( R = 517 \text{ psi} \) was conservatively selected.

Table A2-4

\[ \rho_{min} = 0.0033 < \rho = 0.0106 \quad \therefore \text{ok} \]

Step 5 Find the required amount of steel:

\[ A_s = \rho bd \]

\[ A_s = 0.0106(12)(27.5) = 3.29 \text{ in}^2 \]

From Table A2-9

Try 3 #10 \( (A_s = 3.81 \text{ in}^2) \)

The reinforcement is placed at the top of the beam, because the moment is negative at the section under investigation, which causes tension at the top. Figure 2-47c shows a sketch of the beam.

Table A2-8

\[ h_{min} = 10.5 \text{ in.} < 12 \text{ in.} \]

### FIGURE 2-47c Sketch of beam for Example 2-12.

Step 6 Check for the actual effective depth, \( d \):

\[ \bar{y} = 1.5 \text{ in.} + \frac{3}{8} \text{ in.} + \frac{1.27}{2} = 2.51 \text{ in.} \]

Cover Stirrup Bar diameter

\[ d = h - y = 30 \text{ in.} - 2.51 \text{ in.} = 27.49 \text{ in.} \equiv d_{\text{assumed}} = 27.5 \text{ in.} \quad \therefore \text{ok} \]
Design example 2

Determine the required area of steel for a reinforced concrete rectangular beam subject to a total factored moment, $M_s = 400$ ft-kips, that already includes the estimated weight of the beam. $f'_c = 4$ ksi, $f_y = 60$ ksi and $f = 0.0124$ from Table 2-1.

Solution

Step 1

$M_s = 400$ ft-kips

Step 2

For $f'_c = 4$ ksi, $f_y = 60$ ksi and $f = 0.0124$

using Table A2-4b $\rightarrow R = 596$ psi

Steps 3 & 4

Search now for the beam's sizes:

$bd^2 = \frac{12,000M_s}{R} = \frac{12,000 \times 400}{596} = 8,054$ in$^3$

There are an infinite number of solutions, that is, an infinite number of concrete cross sections that will satisfy the design problem, even with the provision that $f = 0.0124 (1.24\%)$. The table below lists a few solutions. Take your pick!

<table>
<thead>
<tr>
<th>$b$ (in)</th>
<th>10</th>
<th>12</th>
<th>14</th>
<th>16</th>
<th>18</th>
<th>20</th>
</tr>
</thead>
<tbody>
<tr>
<td>$d$ (in)</td>
<td>28.4</td>
<td>26.0</td>
<td>24.0</td>
<td>22.5</td>
<td>21.2</td>
<td>20.1</td>
</tr>
<tr>
<td>$A_{(xx)}$ (in$^2$)</td>
<td>3.52</td>
<td>3.87</td>
<td>4.17</td>
<td>4.46</td>
<td>4.73</td>
<td>4.98</td>
</tr>
<tr>
<td>$h_{(xx)}$ (in)</td>
<td>12</td>
<td>13</td>
<td>28</td>
<td>28</td>
<td>24</td>
<td>24</td>
</tr>
</tbody>
</table>

Another way to solve this same problem is to select a $d/b$ ratio. For example, suppose that after determining that

$bd^2 = 8,054$ in$^3$

the designer selects a $d/b = 2.0$ ratio. Then:

$b = \frac{d}{2}$

$\frac{d}{2} (\frac{d^3}{2}) = \frac{d^3}{2} = 8,054$

$d = 3\sqrt{2} \times 8,054 = 25.3$ in.

$b = \frac{d}{2} = \frac{25.3}{2} = 12.65$ in. $\rightarrow$ Select $b = 13$ in.

$h = 25.3 + 2.5 = 27.8$ in. $\rightarrow$ Select $h = 28$ in.

$A_s = 0.0124 \times 13 \times (28 - 2.5) = 4.11$ in$^2$
T-beams

- **Effective flange width - beams with slab on both sides** (ACI 8.10)
  - \( b_{\text{eff}} < \frac{1}{4} \) span of beam
  - Eff. overhang width on each side < 8 \( h_t \) and \( \frac{1}{2} \) clear distance to next web

- **Effective flange width - beams with slab on one side only** (ACI 8.10)
  - Effective overhang width less than
    - \( 1/12 \) span of beam
    - \( 6 \ h_t \) and \( \frac{1}{2} \) clear distance to next web
T-beams

T-Beam analysis

Assume \( f_s = f_t \)

\[ A_{sf} = 0.85 f_t (b_{eff} - b_w) \frac{A_f}{f_t} \]

\[ (a) \quad (b) \quad (c) \]

T-beams

- Normal resisting moment
  \[ M_{n} = A_{sf} (d - \frac{h_f}{2}) \]

- Remaining area of steel \((A_s - A_{sf})\)
  Depth of equivalent rectangular stress block in the web of the beam
  \[ a = \frac{(A_s - A_{sf}) f_t}{0.85 f_t b_w} \]

- Additional resisting moment
  \[ M_{n+} = (A_s - A_{sf}) f_t \left( d - \frac{h_f}{2} \right) \]

Asf will balance the longitudinal compressive force in the overhanging portions of the flange. (stressed to 0.85f)

Page 47
T-beams

Total normal moment capacity

\[ M_n = M_{n1} + M_{n2} = Af_y \left( d - \frac{bl_1}{e} \right) + (b - Af_y) f_y (d - \frac{b}{2}) \]

Ensure that \( \varepsilon_s \geq 0.004 \)

From geometry

\[ \frac{s}{d_1} \leq \frac{2}{6} + \frac{0.002}{0.002 + 0.004} = 0.469 \]

So

\[ f_s = \frac{A_s}{b d} \quad \text{to} \quad f_{n,\text{max}} = f_{n,\text{max}} + \frac{f_t}{A_t} \]

For \( \varepsilon_s = 0.005 \rightarrow s = 0.375 \)

Minimum tensile reinforcement ratio

\[ f_{\text{min}} = \max \left\{ \frac{5}{15} \frac{f_t}{f_y}, \frac{s}{d} \right\} \]